CIRCULATION TIME DISTRIBUTION IN A MIXED CHARGE*

V.KUDRNA, I.FOŘT, J.VLČEK and J.DRAŽIL

Department of Chemical Engineering,
Institute of Chemical Technology, 166 28 Prague 6

Received June 25th, 1973

The frequency distribution function is proposed for the circulation time of a tracer particle in a mechanically mixed charge on the basis of a simple stochastic model. It has been confirmed that this function describes the experimental results with an accuracy commonly required for similar operations.

Tracer particle, *i.e.* a solid, visually or otherwise easily observable[1-3] body following sufficiently closely the motion of the liquid, is often used for the study of liquid flow in a mechanically mixed charge. The tracer particle method offers convenient means of determining *e.g.* the circulation time of liquid in a charge characterized as a period between two consecutive passages of the tracer particle through the rotor region of the impeller[4].**

The quantity most often determined is the mean circulation time, $\overline{\Delta t}$, evaluated from

$$\overline{\Delta t} = (1/N) \sum_{i=1}^{N} t_1 . \qquad (1)$$

The knowledge of this quantity will do for the solution of many problems; it enables one for example to determine the pumping capacity of the impeller, *i.e* the volume of the liquid passing through the rotor region per unit time[2-5].

In description of other phenomena, *e.g.* for an estimate of the time of homogenation of two miscible liquids, one needs a more detailed characteristic: the frequency or the distribution function of the circulation time[6-8]. Clearly, such function provides a far more thorough information about the system in question and may be useful also for a more profound theoretical study of liquid flow in a mixed batch. Experimental determination and evaluation of the frequency function have been examined in several papers[3,4,6,9]. Yet, none of the authors found an analytical expression of the frequency function describing satisfactorily the experimental data.

* Part XL in the series Studies on Mixing; Part XXXIX: This Journal *39*, 1810 (1974).

** The term rotor region refers to the cylindrical volume encompassing the rotating impeller. This hypothetical cylinder is coaxial with the impeller; its diameter and height equal respectively the diameter of the impeller and the height of the blades.

Fořt[4], for instance, approximated the function by an exponential. It must be noted, however, that approximations of this kind are satisfactorily accurate only for large values of the circulation time. A combination of a straight line a sinusoid and an exponential curve proposed by O'Shima and Yuge[9] contains eight empirical parameters and it is thus impractical.

THEORETICAL

To find a suitable form of the frequency function a model has been proposed which though severely abstracted from the real situation enables a simple expression of the frequency distribution to be found. The model is based on the following simplifying assumptions:

A 1) Consider an infinite space filled with liquid containing in the origin a point source of momentum, *i.e.* the impeller the dimensions of which are neglected. *A 2*) The motion of the tracer particle of infinitesimal size is isotropic, *i.e.* a function of the distance from the origin only. *A 3*) Having passed through the source of momentum the tracer particle is ejected at some instant t in such a manner that it reaches at the same instant the distance r_0 from the origin. *A 4*) The tracer particle approaches the origin at a velocity the radial component of which is proportional to the distance from the origin. *A 5*) The source of momentum causes random fluctuations of the tracer particle superimposed on the motion described in assumption *A 4*). The fluctuations are function of time and proportional to the unidimensional Wiener's process.* *A 6*) The phenomenon described by the above assumptions repeats on subsequent passages of the tracer particle through the origin.

It is apparent that the position of the tracer particle in accord with the assumption *A 2*) is fully determined by a single dimension – the distance from the origin. Simultaneously, this quantity is a random function of time, denoted by $R(\Theta)$. The motion of the particle may be described by the following stochastic differential equation in which the right hand side terms follow from the assumptions *A 4* and *A 5* (ref.[11]):

$$\mathrm{d}R(\Theta) = -\omega R(\Theta)\,\mathrm{d}\Theta + \sigma\omega^{1/2}\,\mathrm{d}W(\Theta)\,. \tag{2}$$

$W(\Theta)$ denotes the Wiener's process and ω and σ are constants for a given intensity of the source of momentum.

From Eq. (*2*), as may be shown[12], follows the Kolmogoroff's forward-difference equation

$$\frac{\partial f(r, \tau; r_0, t)}{\partial \tau} - \omega\,\frac{\partial r f(r, \tau; r_0, t)}{\partial r} - \frac{1}{2}\,\omega\Theta^2\,\frac{\partial^2 f(r, \tau; r_0, t)}{\partial^2 r} = 0\,. \tag{3}$$

* By Wiener's process it is understood a normally distributed random function of time, zero mean and dispersion equal the time elapsed from the beginning of the process[10].

Here, $f(r, \tau; r_0, t)$ is the transitive probability density defined by

$$\frac{\partial f(r, \tau; r_0, t)}{\partial r} = F(r, \tau; r_0, t), \quad [t \leqq \tau], \tag{3a}$$

where the distribution function $F(r, \tau, r_0, t)$ expresses the probability that the distance of the tracer particle from the origin at time $\Theta = \tau$ will be smaller than r assuming that at the instant $\Theta = t$ the particle was at r_0 (see assumption *A 3*).*

TABLE I

Experimental Frequency Function of the Circulation Time

D/d	n s^{-1}	ω s	ϱ	$\overline{\Delta t}$ s	Result of χ^2 test	$K_1 \cdot 10^{-1}$	K_2	$Re \cdot 10^{-4}$
				Paddle				
3	5·85	0·38	1·39	3·74	1	0·65	2·20	5·85
3	8·35	0·54	1·33	2·56	1	0·64	2·13	8·35
3	10·8	0·65	1·23	2·03	1	0·60	2·04	10·8
4	10·0	0·27	1·18	4·78	1	0·27	2·22	5·62
4	11·3	0·36	1·22	3·57	0	0·27	2·26	7·50
4	16·7	0·45	1·18	2·83	1	0·27	2·21	9·35
5	15·0	0·16	0·97	6·94	1	0·11	2·17	5·40
5	2·16	0·23	0·93	4·73	1	0·11	2·11	7·80
5	28·4	0·33	0·94	3·35	1	0·12	2·10	10·0
				Turbine				
3	2·5	0·15	1·29	9·13	1	0·59	2·10	2·50
3	3·33	0·20	1·20	6·30	1	0·61	1·99	3·33
3	4·16	0·24	1·19	5·34	1	0·58	1·99	4·16
3	5·0	0·24	1·11	5·06	1	0·48	1·90	5·00
4	5·0	0·10	1·03	11·5	1	0·20	2·04	2·81
4	6·67	0·13	1·02	8·63	1	0·20	2·02	3·75
4	8·35	0·16	1·00	7·35	1	0·19	2·00	4·68
				Propeller				
4	11·7	0·19	0·95	5·91	1	0·16	1·76	3·28[a]
4	16·7	0·33	1·01	3·59	1	0·20	1·82	9·35
4	16·7	0·30	0·97	3·82	1	0·18	1·76	1·04[b]
4	20·0	0·39	1·21	3·34	1	0·19	2·02	5·62[a]

[a] Aqueous solution of glycerol; viscosity 2 cP; [b] aqueous solution of glycerol; viscosity 9 cP.

* It may be shown that the fundamental solution of Eq. (*3a*) provides a non-zero probability for the event that the tracer particle appears in the origin of the system. This is a consequence of the fact that the equation describes the motion essentially as a unidimensional one.

Thus we have

$$F(r, \tau, r_0, t) = P\{R(\tau) < r/R(t) = r_0\}, \quad [t \leqq \tau] . \tag{3b}$$

Consider now a situation when the tracer particle passes the origin for the first time since the time t. Clearly, such instant $\Theta = T$ is a random variable. In view of the assumption *A 3* and *A 6* the difference $\Delta T = (T - t)$ may be though of as the circulation time which is also a random variable. The probability that ΔT is smaller than the interval Δt is

$$\Psi(\Delta t) = P\{\Delta T < \Delta t\} \tag{4}$$

and the corresponding probability density

$$\psi(\Delta t) = \mathrm{d}\Psi(\Delta t)/\mathrm{d}\Delta t . \tag{5}$$

This function may be found[13] provided that the solution of Eq. (3), $f(r, \tau, r_0, t)$, is known, for the initial condition following from the assumption *A 3*: $f(r, \tau, r_0, t)_{\tau=t} = \delta(r_0 - r)$ (the symbol δ on the right hand side denotes the Dirac function) and the boundary conditions following from the assumption *A 1* and *A 3*:

$$f(r, \tau, r_0, t)|_{r=0} = f(r, \tau, r_0, t)|_{r\to\infty} = 0 , \quad [t < \tau] .$$

Then we have

$$f(r, \tau, r_0, t) = \frac{1}{(2\pi\varkappa^2)^{1/2}} \left\{\exp\left[-\frac{(r-\bar{r})^2}{2\varkappa^2}\right] - \exp\left[-\frac{(r+\bar{r})^2}{2\varkappa^2}\right]\right\}, \tag{6}$$

where $\bar{r} = r_0 \exp[-\omega(\tau - t)]$, $\varkappa^2 = (\sigma^2/2)\{1 - \exp[-2\omega(\tau - t)]\}$. The functions expressed in Eqs (5) and (6) are related by [13]

$$\psi(\Delta t) = -\frac{\partial}{\partial \tau}\int_0^\infty f(r, \tau, r_0, t)\,\mathrm{d}r\Big|_{\tau=t+\Delta t} = -\frac{\partial}{\partial \tau}\Phi\left(\frac{\bar{r}}{(2\pi)^{1/2}\sigma}\right)\Bigg|_{\tau=t+\Delta t} =$$

$$= \frac{2\varrho\omega}{\pi^{1/2}} \frac{\exp(-\omega\,\Delta t)}{[1 - \exp(-2\omega\,\Delta t)]^{3/2}} \exp\left[-\varrho^2 \frac{\exp(-2\omega\,\Delta t)}{1 - \exp(-2\omega\,\Delta t)}\right], \tag{7}$$

where $\Phi(x) = \int_0^x \exp(-2y)\,\mathrm{d}y$, $\varrho = r_0/(2\sigma^2)^{1/2}$.

The frequency function has the same form as the probability density $\psi(\Delta t)$. The values of the parameters ϱ and ω are replaced by their estimates obtained from the experimental data.

In addition one can find the corresponding expected value of the circulation time from the relation

$$E(\Delta T) = \int_0^\infty \Delta t \, \psi(\Delta t) \, \mathrm{d}\Delta t \,, \tag{8}$$

whose experimental estimate is the mean defined by Eq. (*1*). The integral cannot be expressed by elementary functions but it is apparent that the expression

$$\omega \, E(\Delta T) = (2\varrho/\pi^{1/2}) \int_0^\infty z \frac{\exp(-z)}{[1-\exp(-2z)]^{3/2}} \exp\left[-\varrho^2 \frac{\exp(-2z)}{1-\exp(-2z)}\right] \mathrm{d}z \,, \tag{9}$$

where $z = \omega \, \Delta t$, is a function of the parameter ϱ only.

EXPERIMENTAL

The experiments designed to test Eq. (7) were performed in a cylindrical perspex glass vessel 0·29 m in diameter equipped with four radial baffles. The following types of impellers were used: A six-blade turbine with flat blades, a six-paddle impeller with flat blades inclined at 45° (ref.[5]) and a three-blade propeller[4].

The vessel-to-impeller diameter ratio, D/d, was varied between 3 and 5. The frequency of revolution, n, ranged from 300 to 1 700 $\min^{-1}$. The batch was distilled water at 20°C held constant to within ±1°. Aqueous solutions of glycerol served as a batch in three preliminary experiments (Table I).

The tracer particle was made of PVC and consisted of three 6 mm mutually perpendicular discs. Its motion was observed visually and the time intervals between two consecutive passages through the rotor region were recorded by a Tesla BM 445 E electronic counter. A detailed

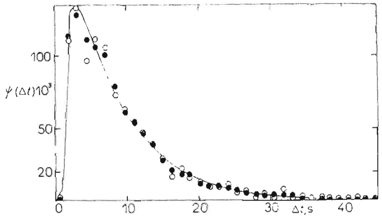

FIG. 1

The Frequency Function

Experimental conditions: Turbine impeller, $D/d = 3$, $n = 2{\cdot}5 \, \mathrm{s}^{-1}$; — theoretical curve, ● smoothed experimental data, ○ experimental data.

description of the particles as well as the experimental routine has been described earlier[1,4]. About 1000 circulation times were recorded in each experimental run.

The parameters ω and ϱ were estimated for each experimental conditions using the maximum likelihood method[14] consisting of the search of the maximum of the logarithm of the probability function defined by

$$L(\omega, \varrho \mid \Delta t_1, \Delta t_2, \ldots, \Delta t_N) = \prod_{i=1}^{N} \psi(\Delta t_i, \omega, \varrho) . \qquad (10)$$

As a next step the average $\overline{\Delta t}$ was calculated from Eq. (*1*). The found values of these quantities for various experimental conditions are summarized in Table I.

In order that we might judge whether the experimental circulation times fit the frequency function (*7*) the chi-square goodness-of-fit test[15] was applied. The data were grouped into classes formed by equidistant intervals chosen so as to keep the number of the classes equal about 25 to 40. To suppress the random effects of grouping a relatively limited set of data the frequencies in individual classes were smoothed using the method of orthogonal polynomials[16]. The critical chi-square values were taken at the 95% significance level. The results of the test are shown in Table I and marked by zero in cases when the hypothesis of the identical experimental and the proposed frequency function was rejected; positive results of the test are marked by unity. A typical example of the theoretical and the experimental frequency function is shown in Fig. 1.

The dependence of the parameter ω on the frequency of revolution and the geometrical arrangement was examined by introducing two dimensionless parameters: $K_1 = \omega/n$, $K_2 = \omega \overline{\Delta t} \times \times (D \cdot d)^{0.4}$. Their numerical values are tabulated also in Table I.

DISCUSSION

It is apparent that the simplifying assumption listed in part Theoretical describes situation considerably different from that existing in the mixing equipment. The proposed relation (Eq. (*7*)) must therefore be regarded as a semi-empirical one where the parameters ϱ and ω possess no clear physical meaning. On the other hand it should be realized that an alternative model leading to a more realistic description of the situation would necessarilly result in excessively complex relations and could not be probably in analytical form.

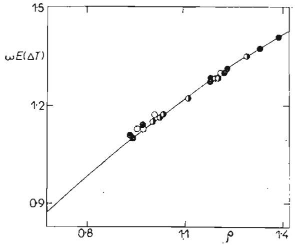

FIG. 2

The Relation between ϱ, ω and the Expected Circulation Time $E(\Delta T)$

— Theoretical curve, experimental data: ● paddle impeller, ◑ turbine, ○ propeller.

From the results subjected to the chi-square goodness-of-fit test it follows, however, that in only one out of twenty examined cases the hypothesis of the identity of the experimental and the theoretical frequency must be rejected and hence that Eq. (7) describes the distribution of the circulation times quite satisfactorily.

For a comparison of the proposed expression with equations of other authors it suffices to note that for Δt large enough to make $1 \gg \exp(-2\omega\,\Delta t)$, the magnitude of ϱ^2 is of the order of 1 and equation (7) transforms into

$$\psi(\Delta t) = 2\varrho\omega/\pi^{1/2} \exp\left[-\omega\,\Delta t\right]. \tag{7a}$$

This suggests that for large Δt the frequency function may be approximated by exponential as done by Fořt[4] and essentially also by O'Shima and Yuge[9].

A quantitative examination of the effect of mixing conditions on the model parameters would require a more detailed and complex experimental study. Yet, certain qualitative conclusions can be drawn immediately. From the values of K_1 given in Table I it may be inferred that ω is directly proportional to the frequency of revolution which is in accord with the findings following from the dimensional analysis. A similar argument leads to the conclusion that ϱ should not depend on the frequency of revolution which has been in fact confirmed by the experiments.

The effect of the vessel/impeller diameter ratio may be assessed from the values of K_2 summarized in Table I. It may be concluded that K_2 does not depend on the conditions of mixing defined as above.

Mutual relation between the parameters ω and ϱ and the mean $\overline{\Delta t}$, obtained by substituting the expected value $E(\Delta T)$ in Eq. (9) by its experimental estimate was also examined. The theoretical relation between $E(\Delta T)$ and ϱ was found numerically and it is shown graphically by curve in Fig. 2. The same figure indicates also the experimental $\omega\,\overline{\Delta t}$ plotted *versus* ϱ found also experimentally. The agreement between the theory and the experiment is very good although it must not be overestimated because all pertaining values were calculated from the same experimental data.

The authors wish to thank Mrs L. Formanová for her assistance in conducting the experiments.

LIST OF SYMBOLS

- D diameter of vessel
- d diameter of impeller
- E expected value (operator)
- F distribution function
- f transitive probability density
- L probability function
- N number of experiments
- n frequency of revolution
- P probability (operator)
- R position of tracer particle

r radial coordinate
r_0 initial position of tracer particle
T instant of passage of tracer particle through origin
ΔT circulation time
t initial time instant
Δt time interval
$\overline{\Delta t}$ mean circulation time
W Wiener's random process
δ Dirac function
Θ time
τ time instant of observation of tracer particle
Φ Laplace function
Ψ distribution function
ψ probability density
Re Reynolds Number

REFERENCES

1. Steidl H.: This Journal *23*, 1644 (1958).
2. Porcelli J. V., Marr G. R.: Ind. Eng. Chem., Fund. *1*, 172 (1962).
3. Marr G. R., Johnson E. F.: A.I.Ch.E.J. *9*, 383 (1963).
4. Fořt I.: This Journal *32*, 3663 (1967).
5. Fořt I., Valešová H., Kudrna V.: This Journal *36*, 164 (1971).
6. Inoue S., Sato K.: Chem. Eng. (Japan) *30*, 922 (1966).
7. Inoue S., Sato K.: Chem. Eng. (Japan) *33*, 293 (1969).
8. O'Shima E., Yuge K.: Chem. Eng. (Japan) *34*, 439 (1970).
9. O'Shima E., Yuge K.: Chem. Eng. (Japan) *33*, 898 (1969).
10. Gichman S. S., Skorochod A. V.: *Stochastičeskije Differencialnyje Uravnenija*, p. 8. Naukova Dumka, Kijev 1968.
11. See 10, p. 33.
12. See 10, p. 100.
13. Svešnikov A. A.: *Prikladnyje Metody Teorii Slučajnych Funkcij*, p. 267. Nauka, Moscow 1968.
14. Himmelblau D. M.: *Process Analysis by Statistical Methods*, p. 50. Wiley, New York 1970.
15. See 14, p. 74.
16. Gust P. G.: *Numerical Methods of Curve Fitting*, p. 349. Univ. Press, Cambridge 1961.

Translated by V. Staněk.